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World vectors, Jacobi vectors and Jacobi one-forms on a manifold with a linear symmetric connection

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Abstract. We solve the Jacobi equation for vector fields and one-forms on a manifold with symmetric linear connection in terms of propagators and discuss their properties. As a generalisation of the concept of position vector, we introduce the world vector field, derive its basic properties and relate it to the Jacobi propagators.

1. Introduction

The subject of this paper is the extension of the concept of two-point vector fields related to the world function from Riemannian manifolds to those which carry a linear symmetric connection.

Ever since the pioneering papers of Ruse (1930, 1931) and Synge (1931), the world function and its derivatives have been used widely in dealing with problems in relativistic theories of gravitation. As long as its applications were restricted to Riemannian spaces, the world function itself could serve as a convenient starting point and basis of all two-point considerations. Relying on the concept of geodesic separation between pairs of world points, it provided immediate access to invariant concepts.

But the notions of pairs of points joined by geodesic curves, parallel propagation of vectors, deviation of neighbouring geodesics and the like are essentially affine concepts, based solely on the existence of geodesic curves on a manifold M, i.e. a linear (symmetric) connection Γ. Thus the extension of world-function-related notions to (M, Γ) undertaken in this paper is a natural procedure, tied directly to previous work (Dixon 1974, 1979, Schattner 1979).

It is even a compelling one: linear connections (which are not derived from a Riemannian metric) on suitable manifolds are useful in the classical mechanics of holonomic systems (Dombrowski and Horneffer 1964). There the configuration space–time is furnished with a linear symmetric connection which is determined by the system (via its kinetic energy tensor) and by the forces (through their potential) in such a way that the possible 'world lines' of the system are geodesic curves. A special case is Newtonian gravitational theory, where space–time is equipped with a Galilei–Newton structure which is determined by the potential of the gravitational field (Cartan 1923, Friedrichs 1928, Trautman 1963, Künzle 1972).

Extensive use of two-point world vectors, their derivatives, and their relation to Jacobi propagators has been made by Dixon (1974, 1979) in his ingenious treatment of the problem of motion of matter. His analysis makes use of the world function which, however, is needed only to derive the world vectors. Though he does not mention it in his writings, he suspected himself that his procedure might be an affine one. That this is indeed true has been noted by Schattner (1981).

The basic concepts that have to be linked are the *exponential map* and the *Jacobi transport of a vector* which can be visualised as the behaviour of a vector joining corresponding points on two neighbouring geodesics. The concept of *Jacobi transport of one-forms* arises naturally from the Euler–Lagrange variation of a simple functional on the cotangent bundle (Trümper 1980). In a Riemannian space it would coincide with the notion of Jacobi transport of a vector. Both versions of Jacobi transport, for vectors and one-forms, give rise to formal solutions of the corresponding second-order DE in terms of *Jacobi propagators* and *Jacobi co-propagators* respectively.

2. Notation and conventions

Unless otherwise stated we follow the notation of Choquet-Bruhat *et al* (1977). Let M be a manifold of class C^∞ equipped with a symmetric linear connection. For a two-point tensor field α the covariant derivative with respect to the first variable is denoted by D and the covariant derivative with respect to the second variable by $\mathbb{D}$. In local coordinates we index the first variable by $i, j, k, \ldots$ and the second by $a, b, c, \ldots$. Furthermore we abbreviate $D \ldots D$ (n times) by D^n, in local coordinates $D_{i_1} \ldots D_{i_n} = D_{i_1 \ldots i_n}$. Using this convention, we suppress the arguments whenever there is no risk of confusion, for example

$$(D^2 \mathbb{D} \alpha(z^i, x^a))_{jklb}^{mc} = D_{jk} \mathbb{D}_b \alpha_l^{mc}.$$

$\langle\alpha\rangle$ denotes the coincidence limit of the bi-tensor field α, i.e. $\langle\alpha\rangle$ is the tensor field defined by $z \mapsto \lim_{x \to z} \alpha(z, x)$.

3. Definition and properties of the Jacobi propagators

Let $u \mapsto x(u)$ be a geodesic† with $x(0) = z$. A unique solution of the equation of geodesic deviation (Jacobi's equation)

$$D_{\dot{x}}^2 J^a + R^a{}_{bcd} \dot{x}^b \dot{x}^c J^d = 0$$

is determined by $J^k(0)$ and $D_{\dot{x}} J^k(0)$. Since the equation is linear, there exist linear maps $K(u)$, $L(u)$ such that

$$J^a(u) = K^a{}_k(u) J^k(0) + L^a{}_k(u) D_{\dot{x}} J^k(0). \tag{3.1}$$

The map $u \mapsto L(u)$ is differentiable in a neighbourhood of 0 and $L(0) = 0$. Therefore there exists a differentiable map $u \mapsto H(u)$ such that $L(u) = uH(u)$. Obviously, H and K depend upon the initial point $z \in M$ and the initial direction $\dot{x}(0)$ of the geodesic $x(u)$, but not on the choice of the affine parameter. Hence it is possible to consider H and K as two-point tensor fields defined in a neighbourhood of the diagonal‡ set of $M \times M$.

† Unless otherwise stated, all geodesics are affinely parametrised.
‡ The diagonal set of $M \times M$ is the set of all pairs (x, x), $x \in M$.

Definition 3.1. The two-point tensor fields K and H are said to be the first and second Jacobi propagators. In analogy we denote the resolvents of the adjoint Jacobi equation,

$$D_{\dot{x}}^2 \iota_a + \iota_d R^d{}_{cba} \dot{x}^c \dot{x}^b = 0,$$

by k and h and call them Jacobi co-propagators.

The following lemmas are easily established.

Lemma 3.1. The coincidence limits of the Jacobi propagators and co-propagators are the respective identities

$$\langle K \rangle|_z = \langle H \rangle|_z = \mathrm{id}_{T_z M}, \qquad \langle k \rangle|_z = \langle h \rangle|_z = \mathrm{id}_{T_z^* M}.$$

Lemma 3.2. The Jacobi propagators K and H are equal to the horizontal and vertical derivatives† of the exponential map.

Let J be a Jacobi field along the geodesic $x(u)$ and let ι be a solution of the adjoint equation (a Jacobi co-vector field). Then the expression

$$J^a D_{\dot{x}} \iota_a - \iota_a D_{\dot{x}} J^a$$

is constant along $x(u)$. This observation may be used to show the following.

Lemma 3.3. There exist the following relations between the propagators and the co-propagators:

$$\begin{pmatrix} D_{\dot{x}}(uh) & -uh \\ D_{\dot{x}} k & -k \end{pmatrix} \circ \begin{pmatrix} K & -uH \\ D_{\dot{x}} K & -D_{\dot{x}}(uH) \end{pmatrix} = \mathrm{id}.$$

The further properties of Jacobi propagators and co-propagators are most easily expressed using their representations in terms of the world vector fields which we are going to introduce in the following section.

4. Definition and properties of the world vector fields

We show that the world vector fields generalise the standard position vectors of $\mathbb{R}^n$ as well as a different concept of position vector which under certain—as we will show—quite restrictive conditions may be introduced on an affine manifold (M, Γ).

Definition 4.1. The two-point world vector fields λ and ρ are defined by

$$\lambda(z, x) = -\exp_z^{-1} x, \qquad \rho(z, x) = \lambda(x, z),$$

in a neighbourhood of the diagonal set of $M \times M$.

If the connection is the Levi–Civita connection of a (pseudo-) Riemannian metric g, then we obviously have the following.

† The terms 'vertical and horizontal derivative' arise naturally in the theory of connections on vector bundles as presented by Dixon (1974). In appendix 2 we shall give a simple explanation following Dixon (1979).

Lemma 4.1. The world vectors are the derivatives of the world function σ, i.e.

$$D\sigma = {}^{\prime}\lambda, \qquad \mathbb{D}\sigma = \rho$$

and

$$2\sigma = g(\lambda, \lambda) = g(\rho, \rho).$$

Therefore we shall use the short-hand notations

$$\sigma^k(z, x) := \lambda^k(z, x) \;\; (= g^{kl}(z) D_l \sigma(z, x)),$$
$$\sigma^a(z, x) := \rho^a(z, x) \;\; (= g^{ab}(x) \mathbb{D}_b \sigma(z, x)).$$

Further, covariant derivatives with respect to z or x are expressed simply by attaching the corresponding indices to the kernel symbol σ.

Let $x(u)$ be a geodesic with $x(0) = z$. Then it is easily seen from the definition that

$$\rho(z, x(u)) = u\dot{x}(u), \qquad \lambda(z, x(u)) = -u\dot{x}(0).$$

This in turn implies the relations

$$\sigma^k \sigma^l{}_k = \sigma^l, \qquad \sigma^k \sigma^a{}_k = \sigma^a, \qquad \sigma^a \sigma^l{}_a = \sigma^l, \qquad \sigma^b \sigma^a{}_b = \sigma^a,$$

which are mere expressions of the fact that ρ and λ are tangent to the geodesic between z and x.

In the case of a Levi-Civita connection, these equations were used to prove a number of useful relations involving the derivatives of the world function. Therefore all those equations remain true for an arbitrary symmetric connection, if one replaces the $(n+1)$th derivative of the world function by the nth derivatives of the world vectors, e.g.

$$\langle \rho \rangle = \langle \lambda \rangle = 0, \qquad \langle D\lambda \rangle = \langle \mathbb{D}\rho \rangle = id, \qquad \langle \mathbb{D}\lambda \rangle = \langle D\rho \rangle = -id, \text{ etc.}$$

Now we show that the world vectors generalise the notion of position vector.

Definition 4.2. A differentiable vector field V is called a position vector field iff
(i) there is a unique point (the 'origin') $0 \in M$ such that $V(0) = 0$;
(ii) for all $x \in M$, $\nabla V(x) = id_{T_x M}$.

It is easily seen that position vectors in standard $\mathbb{R}^n$ are characterised by these properties. In appendix 1 we show that a *Levi-Civita connection* is flat if it admits a position vector field. In order to show in which sense the world vectors generalise the notion of position vector on a manifold (M, Γ) we state the following.

Proposition 4.2. Let (M, Γ) be such that it admits a position vector field V. Then

$$V(x) = \rho(0, x).$$

Proof. The integral curves of V are reparametrised geodesics, since $\nabla_V V = V$. Now, $\nabla V = id$ implies that for all $X \in T_0 M$ there is an integral curve ϕ of V through 0 tangent to X, if properly reparametrised. Therefore

$$V(\exp_0(uX)) = \Omega(u) \frac{\mathrm{d}}{\mathrm{d}u} \exp_0(uX)$$

for some function Ω. Differentiating again, we find

$$\frac{\mathrm{d}}{\mathrm{d}u}\Omega(u)\frac{\mathrm{d}}{\mathrm{d}u}\exp_0(uX)=\frac{\mathrm{D}}{\mathrm{d}u}V(\exp_0(uX))=\frac{\mathrm{d}}{\mathrm{d}u}\exp_0(uX)$$

since $\exp_0(uX)$ is a geodesic and $\nabla V = id$. Therefore

$$\mathrm{d}\Omega/\mathrm{d}u=1 \qquad \text{or} \qquad \Omega=u+C$$

for some constant C. Since $V(0)=0$, we find $\Omega=u$, whence the result.

Corollary 4.3. In standard $\mathbb{R}^n$ we have

$$\rho(z,x)=x-z.$$

5. Jacobi propagators and world vectors

Proposition 5.1. In terms of the world vectors the Jacobi propagators may be expressed as follows:

$$H^a{}_k(z,x)\sigma^k{}_b(z,x)=-\delta^a{}_b(x), \qquad K^a{}_k(z,x)=H^a{}_l(z,x)\sigma^l{}_k(z,x).$$

Proof. Let $x(u,s)$ be a one-parameter family of geodesics such that $x(0,s)=z(s)$. $J:=\partial_s x$ is a Jacobi field along $x(u,s)$. Differentiation of

$$\sigma^k(z(s),x(u,s))=-u\,\partial_u x^k(u,s)|_{u=0}$$

with respect to s yields

$$\sigma^k{}_l J^l+\sigma^k{}_a J^a=-uD_s\partial_u x^k(u,s)|_{u=0}.$$

But

$$D_s\partial_u x(u,s)=D_u\partial_s x(u,s)=D_u J(u).$$

Setting $s=0$, we obtain an equation which holds for all Jacobi fields along $x(u)=x(u,0)$. Inserting the defining equation (3.1) and using the fact that J^k and D_uJ^k are arbitrary, we confirm the assertion.

Corollary 5.2. The Jacobi co-propagators may be expressed as

$$h_a{}^k(z,x)\sigma^a{}_l(z,x)=-\delta^k_l(z), \qquad k_a{}^k(z,x)=h_a{}^l(z,x)\sigma^k{}_l(z,x).$$

Proof. Using in the conserved quantity a Jacobi field such that $J^a(x)=0$, one obtains

$$D_uJ^a\iota_a=D_uJ^k\iota_k-J^kD_u\iota_k.$$

Evaluation of D_uJ^a and D_uJ^k in terms of J^k and the Jacobi propagators (using the fact that $J^a=0$) yields the result.

On the basis of these representations, various properties of the Jacobi (co-) propagators are easily established, e.g.

$$H^a{}_k\sigma^k=-\sigma^a, \qquad h_a{}^k\sigma^a=-\sigma^k, \qquad K^a{}_k\sigma^k=-\sigma^a, \qquad k_a{}^k\sigma^a=-\sigma^k.$$

These equations are related to the fact that the tangent to a geodesic as well as the tangent multiplied by an affine parameter are themselves Jacobi propagated.

6. Outlook

The concepts developed here will be used in the extension of Dixon's laws of motion to a treatment of gravitation which embraces both Newton's and Einstein's theories (Schattner 1981). In another paper, the present authors will treat the behaviour of world-vector-related concepts under a perturbation of the connection.

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Appendix 1.

In this appendix we examine the restriction imposed locally on the geometry of (M, Γ) by the existence of a position vector field.

Lemma A1. Let M be a C^3 manifold with a C^2 linear connection Γ which admits a C^2 position vector field V. Then

$$R^a{}_{bcd}V^b = 0.$$

Proof. $\nabla_{ab}V^c = 0$, Ricci identity.

Remark. The existence of a position vector field V does not, however, imply that (M, Γ) is flat, even when the connection is symmetric and analytic; we give the following example.

Let $M = \mathbb{R}^2$ with standard polar coordinates (r, ϕ). Let Γ be defined by

$$\nabla\partial_r = (1/r)\partial_\phi \otimes \mathrm{d}\phi, \qquad \nabla\partial_\phi = (1/r)\partial_\phi \otimes \mathrm{d}r - r(1+r^4)\partial_r \otimes \mathrm{d}\phi.$$

Then it is easily checked that $r\partial_r$ is a position vector and that Γ is symmetric and analytic, but the Riemann tensor is not zero.

In the case of a Levi-Civita connection, however, the existence of a position vector field is strong enough to ensure flatness.

Proposition A2. Let M be a C^4 manifold with a C^3 (pseudo-) Riemannian metric g, admitting a C^2 position vector field V with respect to the Levi-Civita connection of g. Then (M, g) is flat.

Proof. Choose a frame $\{\underset{\alpha}{E}\}$ of T_0M. By parallel propagation of the $\underset{\alpha}{E}$ along the geodesic rays emerging from 0, this defines a C^2 frame field. Then the $\underset{\alpha}{E}$ satisfy

$$V^b\nabla_b\underset{\alpha}{E}^a = 0.$$

Differentiation and use of the Ricci identity yield

$$V^b\nabla_b\nabla_c\underset{\alpha}{E}^a + \nabla_c\underset{\alpha}{E}^a = -R^a{}_{dbc}\underset{\alpha}{E}^dV^c. \tag{A1.1}$$

Lemma A1 and the (additional) symmetries of the Riemann tensor of a Levi–Civita connection imply that the right-hand side of (A1.1) vanishes. In view of proposition 4.3, equation (A1.1) is then equivalent to

$$D_u(u\nabla_a E^b) = 0 \tag{A1.2}$$

along all geodesics emerging from 0. Since $V(0)=0$, (A1.1) implies that

$$\nabla_a \underset{\alpha}{E}^b(0) = 0,$$

but the only possibility to satisfy both equations is $\nabla_a \underset{\alpha}{E}^b = 0$.

Appendix 2.

Let Φ be a smooth vector field over the natural projection π: $TM \to M$, i.e. Φ is a map of TM into itself such that $\pi \cdot \Phi = \pi$. Let (x^a, X^a) be a chart in a neighbourhood of $\pi^{-1}(z)$, $z \in M$, such that X^a are affine coordinates of T_xM. Then we can define the *vertical covariant derivative* by $\nabla_{*k}\Phi^l = \partial\Phi^l/\partial X^k$, i.e. we keep the base point fixed and move in the fibre using the natural flat connection of $T_xM \cong \mathbb{R}^n$. The corresponding operation—keeping X fixed and moving the base point $z \in M$—is of course not possible since X must be an element of the fibre over z. If M is equipped with a linear connection it is, however, possible to move X by parallel transport as we change z. Let $x(u)$ be a curve such that $x(0)=z$ and let $X(u)$ be defined by $D_uX(u)=0$, $X(0)=X$. Then $\Phi(x(u), X(u))$ is a well defined ordinary vector field along $x(u)$ whose covariant derivative is given as usual by

$$D_u\Phi^k = \mathrm{d}\Phi^k/\mathrm{d}u + \Gamma^k_{lm}\Phi^l\dot{x}^m.$$

Since

$$D_uX^k \equiv \mathrm{d}X^k/\mathrm{d}u + \Gamma^k_{lm}X^l\dot{x}^m = 0$$

by hypothesis, we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\Phi^k = \left(\frac{\partial\Phi^k}{\partial z^l} - \Gamma^m_{nl}\frac{\partial\Phi^k}{\partial X^m}X^n\right)\dot{x}^l$$

such that

$$D_u\Phi^k = \left\{\frac{\partial\Phi^k}{\partial z^l} + \Gamma^k_{ml}\Phi^m - \Gamma^m_{nl}\frac{\partial\Phi^k}{\partial X^m}X^n\right\}\dot{x}^l \equiv \nabla_{l*}\Phi^k\dot{x}^l$$

where $\nabla_{l*}\Phi^k$ is the *horizontal covariant derivative* defined by

$$\nabla_{l*}\Phi^k := \nabla_l\Phi^k - \Gamma^m_{nl}X^n\,\partial\Phi^k/\partial X^m.$$

Clearly this extends to arbitrary tensor fields over π.

These operations occur naturally as horizontal and vertical parts of a total covariant derivative defined on the tangent bundle. It was through this approach that they were first introduced by Dixon (1974).

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